APPLIED MATHS (HIGHER LEVEL)

LEAVING CERTIFICATE 2008

DETAILED SOLUTIONS AND MARKING SCHEME

Question 1

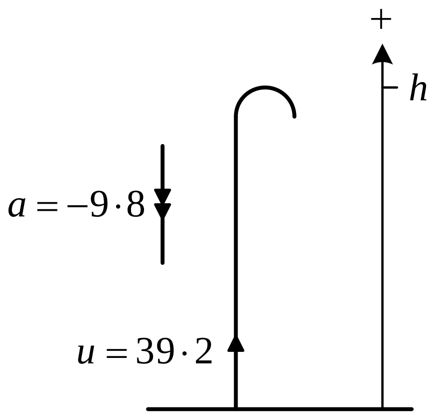
(a) A ball is thrown vertically upwards with an initial velocity of $39 \cdot 2$ m/s.
Find **(i)** the time taken to reach the maximum height
(ii) the distance travelled in 5 seconds.

(b) Two particles P and Q, each having constant acceleration, are moving in the same direction along parallel lines. When P passes Q the speeds are 23 m/s and $5 \cdot 5$ m/s respectively. Two minutes later Q passes P, and Q is then moving at $65 \cdot 5$ m/s.
Find **(i)** the acceleration of P and the acceleration of Q
(ii) the speed of P when Q overtakes it
(iii) the distance P is ahead of Q when they are moving with equal speeds.

Solution

(a) **(i)** Interval: From projection to max height.

$u = 39 \cdot 2$
$v = 0$
$a = -g = -9 \cdot 8$
$s = h$
$t = ?$



Then
$v = u + at$
$0 = 39 \cdot 2 - 9 \cdot 8t$
$9 \cdot 8t = 39 \cdot 2$
$t = 4$ s. **<u>5 marks</u>**

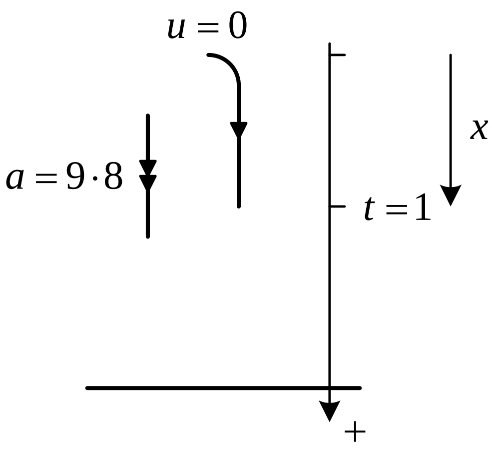
(ii) *(As 5 s is longer than the time to max height, we cannot use the data table above to find the* ***distance*** *travelled in 5 s. We will find the max height, and then the distance travelled in the first second of the downward journey. Finally, we will add these two distances together.)*

$$s = ut + \frac{1}{2}at^2$$
$$h = (39 \cdot 2)(4) + \frac{1}{2}(-9 \cdot 8)(4)^2$$

$h = 78 \cdot 4$ m **5 marks**

(This is the maximum height reached.)

Interval: Down from max height for one second.



$u = 0$
$v =$
$a = g = 9 \cdot 8$
$s = x$
$t = 1$

Then

$$s = ut + \frac{1}{2}at^2$$

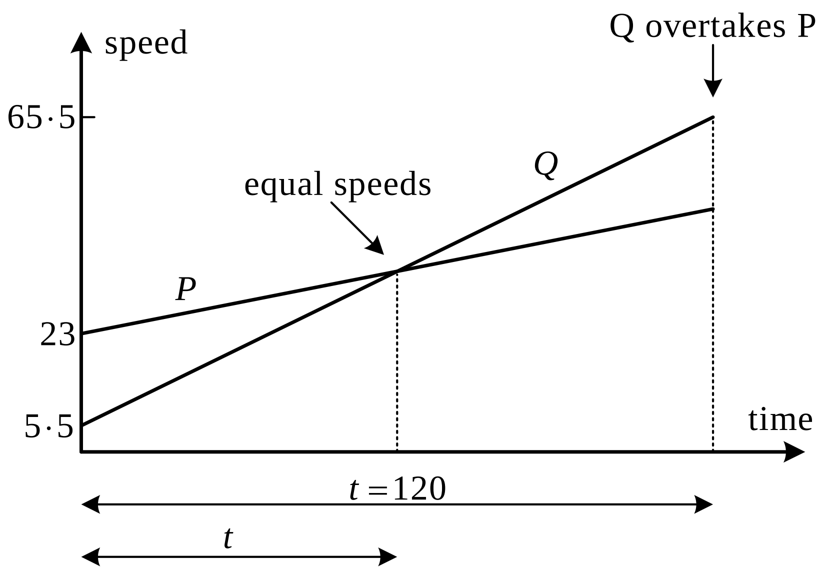
$$x = (0)(1) + \frac{1}{2}(9 \cdot 8)(1)^2$$

$$x = 4 \cdot 9 \text{ m}$$ **5 marks**

The total distance travelled

$= h + x$
$= 78 \cdot 4 + 4 \cdot 9$
$= 83 \cdot 3$ m. **5 marks**

(b) (i)



(We have more information about Q than we have about P. So we will start with Q first and work out its acceleration. When Q overtakes P, the distances travelled are equal. Thus if we find the distance travelled by Q, we can use it for P. This will allow us to find the acceleration of P.)

Q interval: From beginning to when Q overtakes P.

$u = 5\cdot 5$
$v = 65\cdot 5$
$a =$
$s =$
$t = 120$

Then

$$v = u + at$$
$$65\cdot 5 = 5\cdot 5 + a(120)$$
$$60 = 120a$$
$$a = \frac{1}{2}\ \text{m/s}^2$$ **<u>5 marks</u>**

Also

$$s = ut + \frac{1}{2}at^2$$
$$s = (5\cdot 5)(120) + \frac{1}{2}\left(\frac{1}{2}\right)(120)^2$$
$$s = 4260\ \text{m}$$ **<u>5 marks</u>**

P interval: From beginning to when Q overtakes P.

$u = 23$
$v =$
$a =$
$s = 4260$
$t = 120$

Then

$$s = ut + \frac{1}{2}at^2$$
$$4260 = (23)(120) + \frac{1}{2}a(120)^2$$
$$4260 = 2760 + 7200a$$
$$7200a = 1500$$
$$a = \frac{15}{72} = \frac{5}{24}\ \text{m/s}^2$$ **<u>5 marks</u>**

(ii) *(We can use the same data table for P above.)*

$$v = u + at$$
$$v = 23 + \left(\frac{5}{24}\right)(120)$$
$$v = 48\ \text{m/s.}$$ **<u>5 marks</u>**

(iii) P interval: From beginning to when they have equal speeds.

$u = 23$ Then: $v = u + at$

$v = v_P$ $v_P = 23 + \frac{5}{24}t$

$a = \frac{5}{24}$

$s = s_P$

$t =$

Q interval: From beginning to when they have equal speeds.

$u = 5 \cdot 5$ Then: $v = u + at$

$v = v_Q$ $v_Q = 5 \cdot 5 + \frac{1}{2}t$

$a = \frac{1}{2}$

$s = s_Q$

$t =$

Equal speeds:

$$v_P = v_Q$$
$$23 + \frac{5}{24}t = 5 \cdot 5 + \frac{1}{2}t$$
$$17 \cdot 5 = \frac{7}{24}t$$
$$t = 60 \text{ s.}$$

<u>5 marks</u>

(The distance P is ahead of Q is $s_P - s_Q$ at this time.)

$$s = ut + \frac{1}{2}at^2$$
$$s_P = (23)(60) + \frac{1}{2}\left(\frac{5}{24}\right)(60)^2$$
$$s_P = 1755$$

and

$$s_Q = (5 \cdot 5)(60) + \frac{1}{2}\left(\frac{1}{2}\right)(60)^2$$
$$s_Q = 1230$$

Distance between P and Q $= s_P - s_Q$

$= 1755 - 1230$

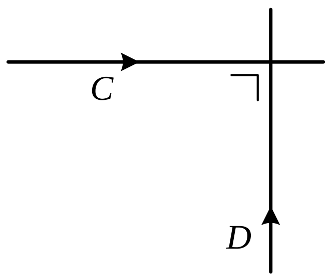
$= 525$ m. **<u>5 marks</u>**

Question 2

(a) Two straight roads cross at right angles. A woman C, is walking towards the intersection with a uniform speed of $1 \cdot 5$ m/s.

Another woman D is moving towards the intersection with a uniform speed of 2 m/s.

C is 100 m away from the intersection as D passes the intersection.



Find

(i) the velocity of C relative to D

(ii) the distance of C from the intersection when they are nearest together.

(b) On a certain day the velocity of the wind, in terms of $\vec{i}$ and $\vec{j}$, is $x\vec{i} - 3\vec{j}$, where $x \in \mathbf{N}$.

$\vec{i}$ and $\vec{j}$ are unit vectors in the directions East and North respectively.

To a man travelling due East the wind appears to come from a direction North $\alpha°$ West where $\tan\alpha = 2$.

When he travels due North at the same speed as before, the wind appears to come from a direction North $\beta°$ West where $\tan\beta = \frac{3}{2}$.

Find the actual direction of the wind.

Solution

(a) **(i)** $\vec{v}_C = 1 \cdot 5\vec{i}$

and

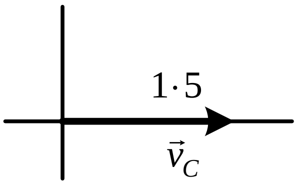
$\vec{v}_D = 2\vec{j}$

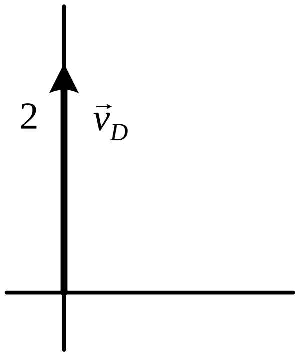
Then

$\vec{v}_{CD} = \vec{v}_C - \vec{v}_D$

$= 1 \cdot 5\vec{i} - 2\vec{j}$

<u>5 marks</u>





From the diagram opposite,

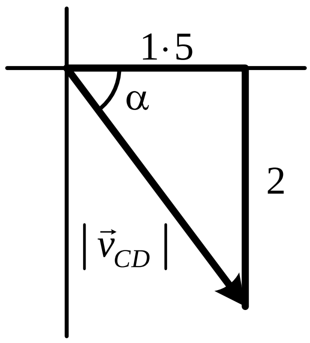
$|\vec{v}_{CD}| = \sqrt{(1 \cdot 5)^2 + (2)^2}$

$= 2 \cdot 5$ m/s **<u>5 marks</u>**

and

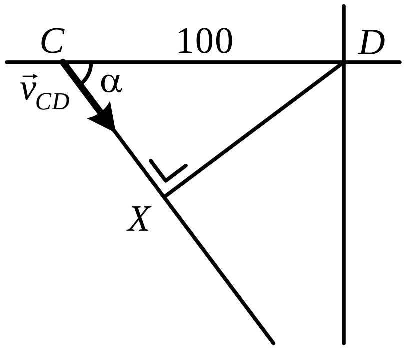
$$\tan\alpha = \frac{2}{1 \cdot 5} = \frac{4}{3}$$

$$\alpha = \tan^{-1}\frac{4}{3} = 53 \cdot 13°$$



Direction $= 53 \cdot 13°$ S of E. **<u>5 marks</u>**

(ii) *(We need to find the time until they are closest together. This is the time taken by C to travel along the relative path from C to X.)*



From the relative path diagram,

$$\frac{|CX|}{100} = \cos\alpha$$

$$|CX| = 100\cos 53{\cdot}13°$$

$$|CX| = 60 \text{ m}$$

Time to nearest approach

$$= \frac{|CX|}{|\vec{v}_{CD}|}$$

$$= \frac{60}{2{\cdot}5}$$

$$= 24 \text{ s}$$

5 marks

In this time, C actually travels

$$1{\cdot}5 \times 24$$

$$= 36 \text{ m}$$

At this time, the distance of C from the intersection is

$$100 - 36$$

$$= 64 \text{ m.}$$

5 marks

(b) Let $\vec{v}_W$ be the velocity of the wind and $\vec{v}_M$ be the velocity of the man.
We are given that

$$\vec{v}_W = x\vec{i} - 3\vec{j}.$$

First scenario: Let the speed of the man be a. This is pointing due east.

Then $\vec{v}_M = a\vec{i}$

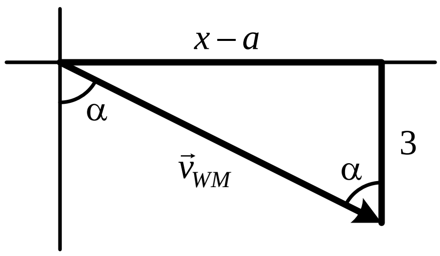
and $\vec{v}_{WM} = \vec{v}_W - \vec{v}_M$

$$= (x\vec{i} - 3\vec{j}) - (a\vec{i})$$

$$= (x - a)\vec{i} - 3\vec{j}$$

5 marks

(As the wind appears to come from α W of N, it is actually travelling in the direction α E of S. The ***lengths*** *of the sides of the triangle are shown opposite.)*



$$\tan\alpha = \frac{x-a}{3}$$

$$2 = \frac{x-a}{3}$$

$x - a = 6$ **... 1** **<u>5 marks</u>**

Second scenario: The velocity of the man is still a, but now it points due north.

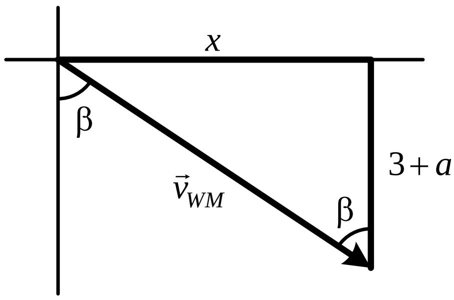
Then $\vec{v}_M = a\vec{j}$

and $\vec{v}_{WM} = \vec{v}_W - \vec{v}_M$

$= (x\vec{i} - 3\vec{j}) - (a\vec{j})$ **<u>5 marks</u>**

$= x\vec{i} - (3+a)\vec{j}$

(This relative velocity points β *E of S. The* ***lengths*** *of the sides of the triangle are shown opposite.)*



$$\tan\beta = \frac{x}{3+a}$$

$$\frac{3}{2} = \frac{x}{3+a}$$

$9 + 3a = 2x$

$-2x + 3a = -9$ **... 2** **<u>5 marks</u>**

Solving **1** and **2**,

1$\times 3$: $3x - 3a = 18$

2: $\underline{-2x + 3a = -9}$

$x = 9$

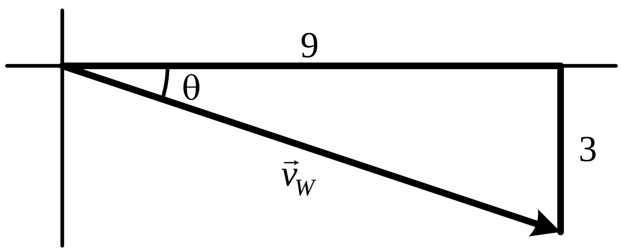
Hence

$\vec{v}_W = 9\vec{i} - 3\vec{j}$

and

$$\tan\theta = \frac{3}{9} = \frac{1}{3}$$

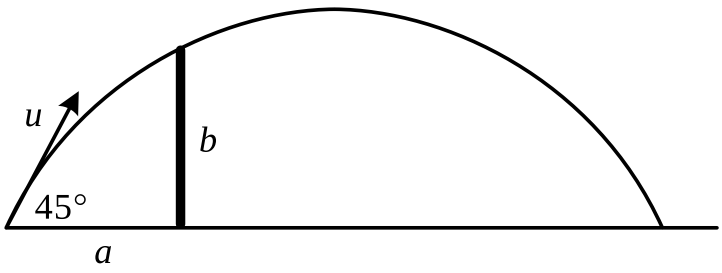
$$\theta = \tan^{-1}\frac{1}{3} = 18\cdot 4^\circ$$



The actual direction of the wind is E $18\cdot 4^\circ$ S, or S $71\cdot 6^\circ$ E. Or we can give the direction the wind comes from as N $71\cdot 6^\circ$ W. **<u>5 marks</u>**

Question 3

(a) A ball is projected from a point on the ground at a distance of a from the foot of a vertical wall of height b, the velocity of projection being u at an angle $45°$ to the horizontal.
If the ball just clears the wall prove that the greatest height reached is $\dfrac{a^2}{4(a-b)}$.



(b) A particle is projected down an inclined plane with initial velocity u m/s.
The line of projection makes an angle of $2\theta°$ with the inclined plane and the plane is inclined at $\theta°$ to the horizontal.
The plane of projection is vertical and contains the line of greatest slope.

The range of the particle on the inclined plane is $\dfrac{ku^2}{g}\sin\theta$.

Find the value of k.

Solution

(a) *(Normally the expression for the greatest height contains u, but here it doesn't. So we will need to write u in terms of a and b. Let v_x and v_y be the horizontal and vertical velocity components respectively. Let s_x and s_y be the horizontal and vertical components of the displacement respectively. The speed of projection is u, and the angle of projection is $45°$. Note that $\cos 45° = \sin 45° = \dfrac{1}{\sqrt{2}}$.)*

$$v_x = u\cos 45° \qquad v_y = u\sin 45° - gt$$

$$v_x = \frac{u}{\sqrt{2}} \qquad v_y = \frac{u}{\sqrt{2}} - gt$$

and

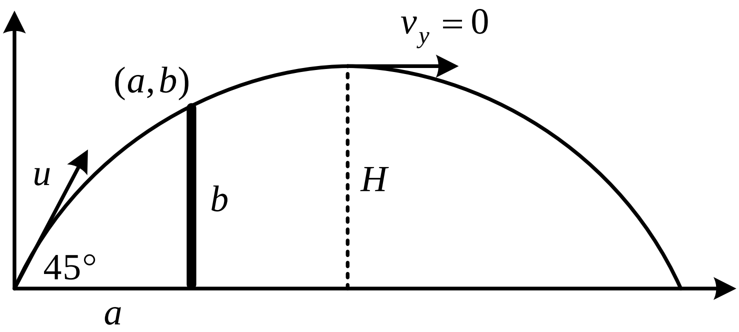
$$s_x = u\cos 45° t \qquad s_y = u\sin 45° t - \frac{1}{2}gt^2$$

$$s_x = \frac{ut}{\sqrt{2}} \qquad s_y = \frac{ut}{\sqrt{2}} - \frac{gt^2}{2}$$

(The path contains the point (a, b). Thus

$$s_x = a \quad \textit{and} \quad s_y = b$$

at the same time.)



$s_x = a:\quad \frac{ut}{\sqrt{2}} = a$ **5 marks**

$$t = \frac{\sqrt{2}\,a}{u} \qquad \textbf{... 1}$$

$s_y = b:\quad \frac{ut}{\sqrt{2}} - \frac{gt^2}{2} = b$ **5 marks**

$$\frac{u}{\sqrt{2}}\left(\frac{\sqrt{2}\,a}{u}\right) - \frac{g}{2}\left(\frac{2a^2}{u^2}\right) = b \qquad \textbf{... from 1}$$

$$a - \frac{ga^2}{u^2} = b$$

$$a - b = \frac{ga^2}{u^2}$$

$$u^2 = \frac{ga^2}{a-b} \qquad \textbf{... 2}$$

(The condition for greatest height is that $v_y = 0$. By putting $v_y = 0$, we find the time to greatest height. Working out s_y at this time gives the actual greatest height.)

$v_y = 0:\quad \frac{u}{\sqrt{2}} - gt = 0$

$$\frac{u}{\sqrt{2}} = gt$$

$$t = \frac{u}{\sqrt{2}\,g}$$ **5 marks**

Max height, *H*, is given by:

$$H = s_y \text{ at } t = \frac{u}{\sqrt{2}\,g}.$$

$$H = \frac{u}{\sqrt{2}}\left(\frac{u}{\sqrt{2}\,g}\right) - \frac{g}{2}\left(\frac{u^2}{2g^2}\right)$$ **5 marks**

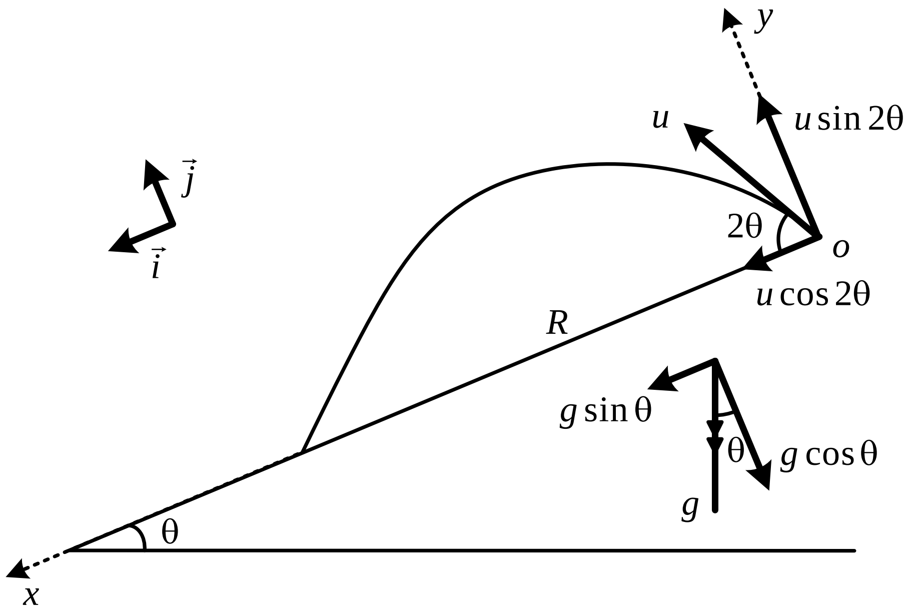
$$H = \frac{u^2}{2g} - \frac{u^2}{4g}$$

$$H = \frac{u^2}{4g}$$

$$H = \frac{1}{4g}\left(\frac{ga^2}{a-b}\right) \qquad \textbf{... from 2}$$

$$H = \frac{a^2}{4(a-b)}. \qquad \underline{\textbf{5 marks}}$$

(b) *(For motion down an inclined plane, we put the x-axis, and thus the $\vec{i}$ vector, pointing down the plane. The y-axis, and $\vec{j}$ vector, point normal to the plane as shown.)*



For this motion,

$$\vec{u} = (u\cos 2\theta)\vec{i} + (u\sin 2\theta)\vec{j}$$

and $$\vec{a} = (g\sin\theta)\vec{i} - (g\cos\theta)\vec{j}.$$

If $\vec{v} = v_x\vec{i} + v_y\vec{j}$ and $\vec{s} = s_x\vec{i} + s_y\vec{j}$ are the velocity and displacement vectors, respectively, then

$$v_x = u\cos 2\theta + g\sin\theta t \qquad v_y = u\sin 2\theta - g\cos 2\theta t$$

and $$s_x = u\cos 2\theta t + \frac{1}{2}g\sin\theta t^2 \qquad s_y = u\sin 2\theta - \frac{1}{2}g\cos\theta t^2.$$

(In this question, we only need s_x and s_y, and these are the only ones we have to write down. To find the range, R, we start by finding the time of flight. The condition for the time of flight is that $s_y = 0$, i.e. the particle lands on the x-axis.)

$s_y = 0$: **<u>5 marks</u>**

$$u \sin 2\theta t - \frac{1}{2} g \cos\theta t^2 = 0$$ **<u>5 marks</u>**

$$u\left(2\sin\theta\cos\theta\right) = \frac{1}{2} g \cos\theta t$$

$$4u\sin\theta = gt$$

$$t = \frac{4u\sin\theta}{g}$$ **<u>5 marks</u>**

(R is s_x at the time of flight.)

$$R = s_x \text{ at } t = \frac{4u\sin\theta}{g}$$

$$R = u\cos 2\theta\left(\frac{4u\sin\theta}{g}\right) + \frac{g}{2}\sin\theta\left(\frac{16u^2\sin^2\theta}{g^2}\right)$$ **<u>5 marks</u>**

$$R = \frac{4u^2}{g}\left(\cos 2\theta\sin\theta\right) + \frac{8u^2}{g}\left(\sin^3\theta\right)$$

$$R = \frac{4u^2\sin\theta}{g}\left[\left(\cos^2\theta - \sin^2\theta\right) + 2\sin^2\theta\right]$$

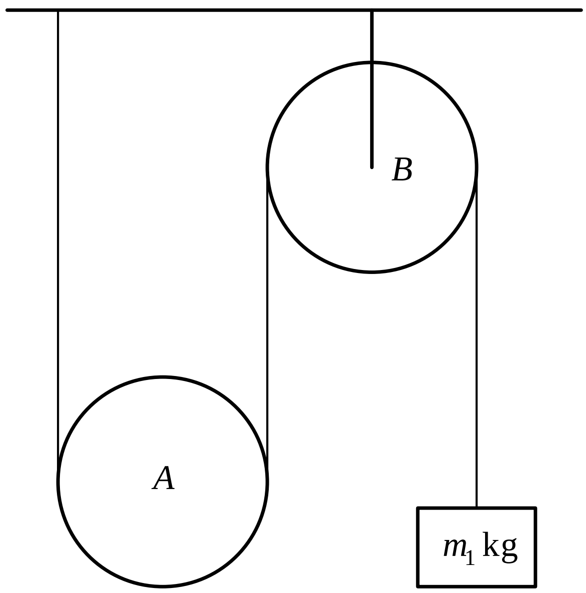
$$R = \frac{4u^2\sin\theta}{g}\left[\cos^2\theta + \sin^2\theta\right]$$

$$R = \frac{4u^2\sin\theta}{g}.$$

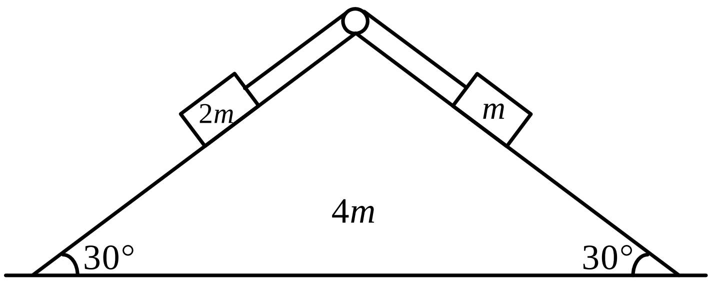
Hence $k = 4$. **<u>5 marks</u>**

Question 4

(a) The diagram shows a light inextensible string having one end fixed, passing under a smooth movable pulley A of mass m kg and then over a fixed smooth light fixed pulley B.
The other end of the string is attached to a particle of mass m_1 kg.
The system is released from rest.
Show that the upward acceleration of A is
$$\frac{(2m_1 - m)g}{4m_1 + m}.$$



(b) Particles of mass $2m$ and m are connected by a light inextensible string which passes over a smooth pulley at the vertex of a wedge-shaped block, one particle resting on each of the smooth faces.
The mass of the wedge is $4m$ and the inclination of the each face to the horizontal is $30°$.



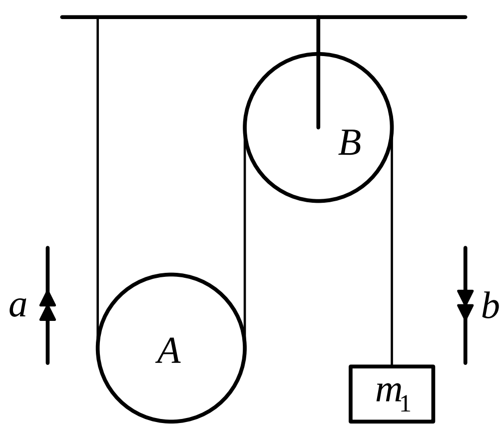
The wedge rests on a smooth horizontal surface and the system is released from rest.

(i) Show, on separate diagrams, the forces acting on the wedge and on the particles.

(ii) Find the acceleration of the wedge.

Solution

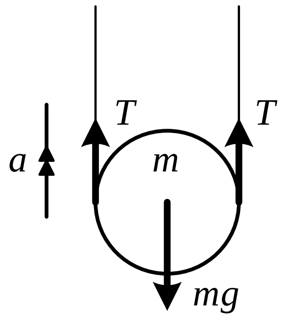
(a) *(If we let a be the upward acceleration of the pulley and b be the downward acceleration of the mass, then b is twice a, as can be seen by discussing distances travelled.)*



$$b = 2a \qquad \textbf{... 1}$$

<u>5 marks</u> **<u>5 marks</u>**

(The forces acting on pulley A are shown opposite. We can now use Newton's 2nd Law on the pulley.)

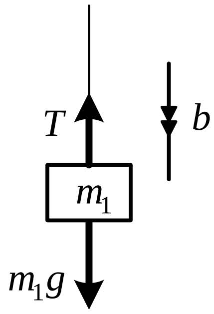


Pulley A:

$(m)(a) = 2T - mg$ **5 marks**

$ma = 2T - mg$ **... 2**

(The next diagram shows the forces acting on the mass. Again we can use Newton's 2nd Law on this mass.)



Mass m_1 :

$(m_1)(b) = m_1 g - T$ **5 marks**

$m_1(2a) = m_1 g - T$ **... from 1**

$2m_1 a = m_1 g - T$ **... 3**

(Now we solve **2** *and* **3** *for a.)*

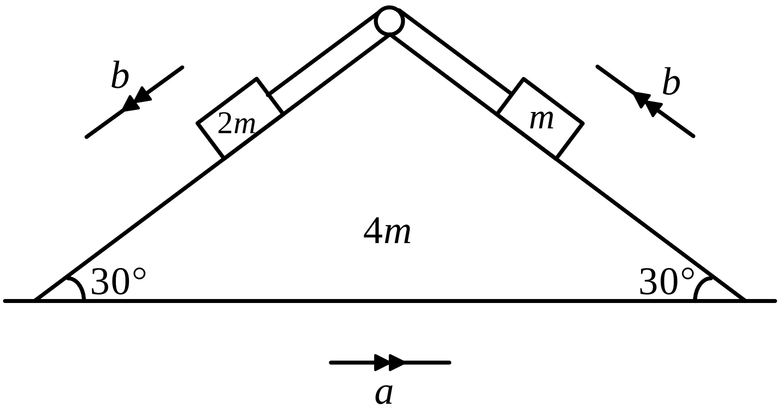
2: $ma = 2T - mg$

3$\times 2$: $\underline{4m_1 a = 2m_1 g - 2T}$

$(4m_1 + m)a = (2m_1 - m)g$ **5 marks**

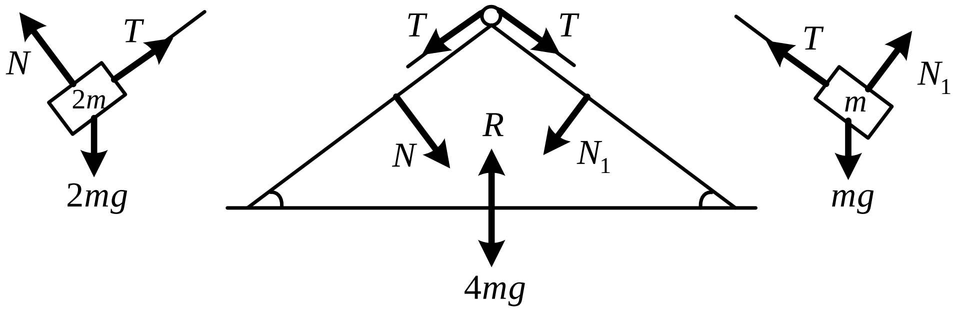
$a = \dfrac{(2m_1 - m)g}{4m_1 + m}.$ **5 marks**

(b) (i) *(Let the horizontal acceleration of the wedge be a, and let the common accelerations of the particles be b.)*



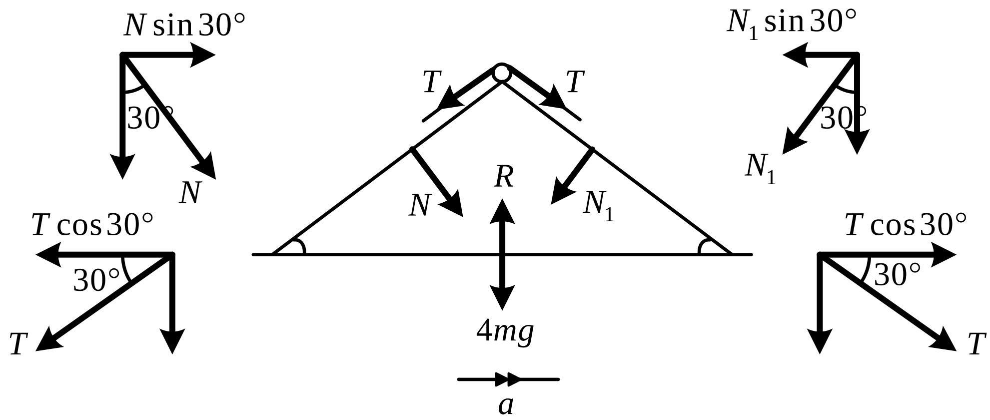
(The diagrams over the page show the forces acting on each of the wedge and the particles. The accelerations are also shown, to facilitate writing down Newton's 2nd Law, but are not required to get the marks for part (i). Nor are the diagrams which show the forces being resolved into components.)

Forces acting on the wedge and the particles:



5 marks **5 marks**

(ii) Wedge: *(We only need to write down the horizontal equation for the wedge.)*

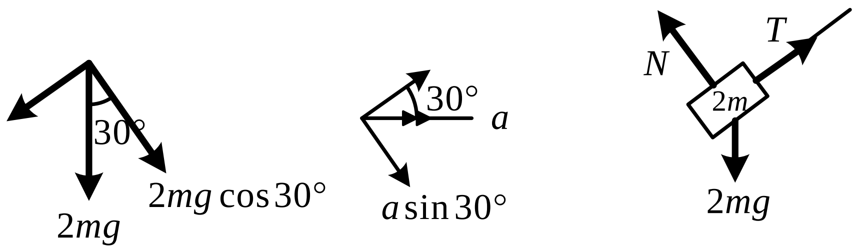


$$\leftrightarrow: \quad (4m)(a) = T\cos 30° - T\cos 30° + N\sin 30° - N_1 \sin 30°$$

$$4ma = \frac{N}{2} - \frac{N_1}{2}$$

$$8ma = N - N_1 \qquad \textbf{... 1}$$

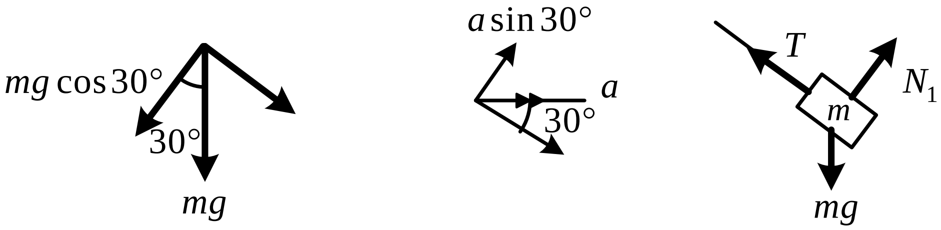
2*m* mass: *(We only need to write down the equation in the direction perpendicular to the face of the wedge.)*



$$\searrow: \quad (2m)(a \sin 30°) = 2mg\cos 30° - N$$

$$ma = \sqrt{3}\, mg - N \qquad \textbf{... 2}$$

***m* mass:** *(We only need to write down the equation in the direction perpendicular to the face of the wedge.)*



⤢ : $(m)(a\sin 30°) = N_1 - mg\cos 30°$

$$\frac{1}{2}ma = N_1 - \frac{\sqrt{3}\,mg}{2} \qquad \textbf{... 3}$$

<u>5 marks</u> (for equations **2** and **3**)

Solving:

2: $N = \sqrt{3}\,mg - ma$

3: $N_1 = \frac{1}{2}ma + \frac{\sqrt{3}\,mg}{2}$

1: $8ma = \left(\sqrt{3}\,mg - ma\right) - \left(\frac{1}{2}ma + \frac{\sqrt{3}\,mg}{2}\right)$

$$\frac{19}{2}ma = \frac{\sqrt{3}}{2}mg$$

$$a = \frac{\sqrt{3}\,g}{19}.$$

<u>5 marks</u>

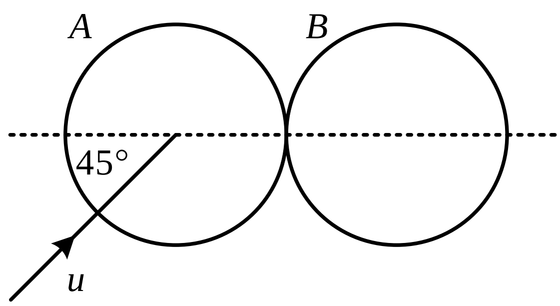
Question 5

(a) Three identical smooth spheres lie at rest on a smooth horizontal table with their centres in a straight line. The first sphere is given a speed of 2 m/s and it collides directly with the second sphere. The second sphere then collides directly with the third sphere.
The coefficient of friction for each collision is *e*, where $e < 1$.

(i) Find, in terms of *e*, the speed of each sphere after two collisions have taken place.

(ii) Show that there will be at least one more collision.

(b) A smooth sphere A moving with speed *u*, collides with an identical smooth sphere B which is at rest.
The direction of motion of A, before impact, makes an angle of $45°$ with the line of centres at the instant of impact.
The coefficient of restitution between the spheres is *e*.

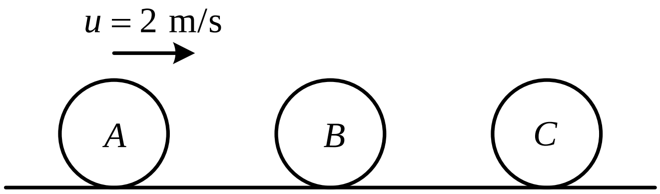


Show that the direction of motion of A is deflected through an angle α where

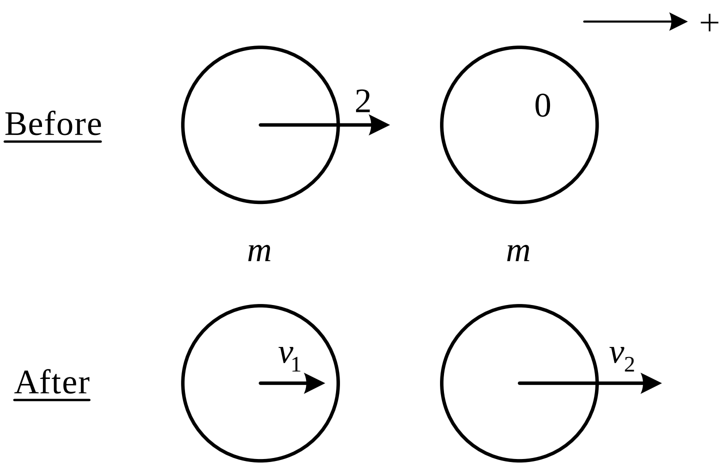
$$\tan\alpha = \frac{1+e}{3-e}.$$

Solution

(a) **(i)** Call the spheres A, B and C and let the mass of each sphere be *m*.



1st Collision (A – B):



Let v_1 and v_2 be the velocities of A and B respectively after the first collision.

PCM: $mv_1 + mv_2 = m(2) + m(0)$ **5 marks**

$v_1 + v_2 = 2$ **... 1**

NEL: $v_1 - v_2 = -e(2-0)$

$v_1 - v_2 = -2e$ **... 2**

Solving:

1: $v_1 + v_2 = 2$

2: $\underline{v_1 - v_2 = -2e}$

$2v_1 \quad = 2 - 2e$

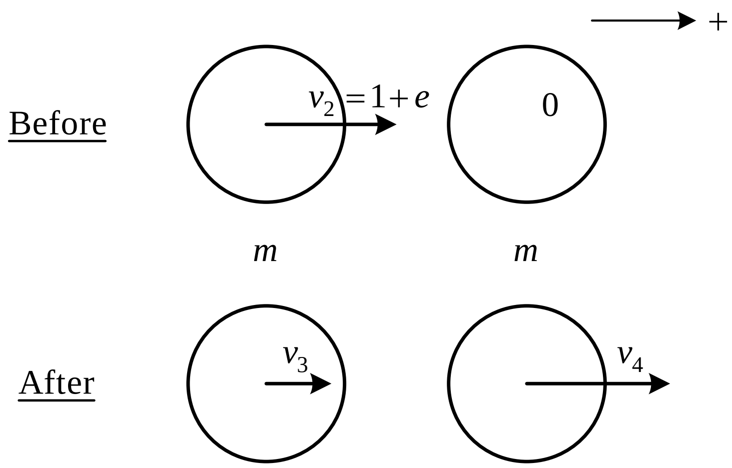
$v_1 = 1 - e$

Then

1: $(1+e) + v_2 = 2$

$v_2 = 1 - e.$ **5 marks**

2nd Collision (B – C):



Let v_3 and v_4 be the velocities of B and C respectively after the second collision.

PCM: $mv_3 + mv_4 = m(1+e) + m(0)$ **5 marks**

$v_3 + v_4 = 1 + e$ **... 3**

NEL: $v_3 - v_4 = -e(1+e-0)$

$v_3 - v_4 = -e - e^2$ **... 4**

Solving:

3: $v_3 + v_4 = 1 + e$

4: $\underline{v_3 - v_4 = -e - e^2}$

$2v_3 \quad = 1 - e^2$

$$v_3 = \frac{1}{2}(1-e^2)$$

Then

3: $$\frac{1}{2}(1-e^2) + v_4 = 1+e$$

$$1-e^2 + 2v_4 = 2+2e$$

$$2v_4 = 1+2e+e^2$$

$$v_4 = \frac{1}{2}(1+e)^2$$

5 marks

Velocities after collisions:

A: $v_1 = 1-e$

B: $v_3 = \frac{1}{2}(1-e^2)$

C: $v_4 = \frac{1}{2}(1+e)^2$.

(ii) There will be a second collision (between A and B) if

$$v_1 > v_3$$

$$1-e > \frac{1}{2}(1-e^2)$$

$$2-2e > 1-e^2$$

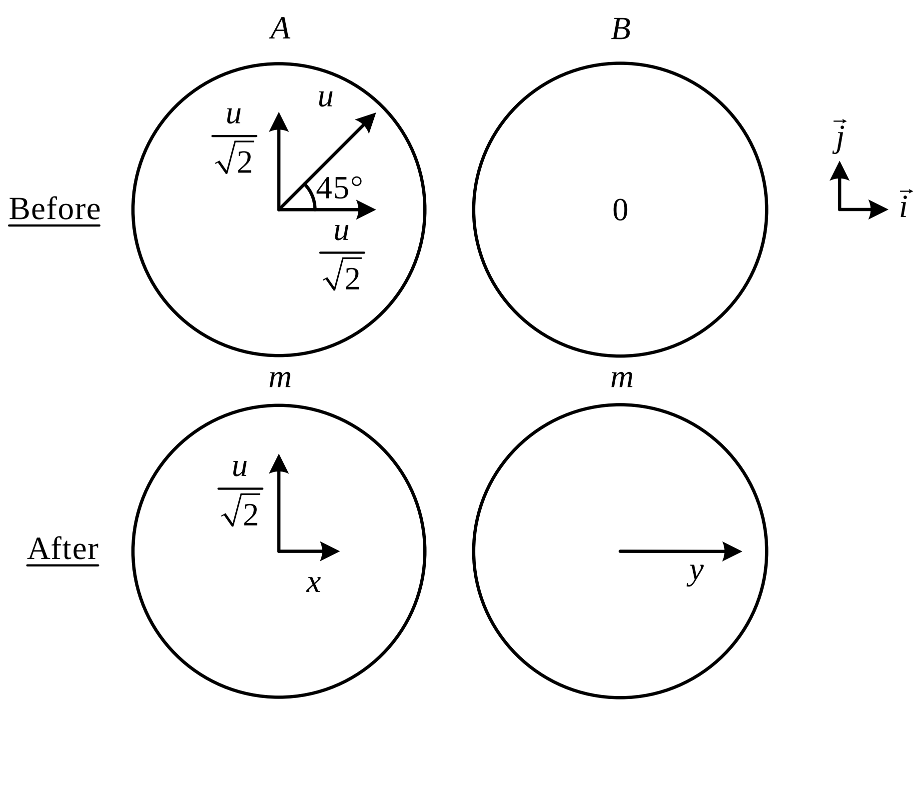
$$e^2 - 2e + 1 > 0$$

$$(e-1)^2 > 0$$

which is true, as $e < 1$.

5 marks

(b) Let the masses of A and B be m. Let x and y be the $\vec{i}$ components, respectively, of the velocities of A and B after the collision.



(We only need to find x for this question.)

PCM($\vec{i}$): $\quad mx + my = m\left(\dfrac{u}{\sqrt{2}}\right) + m(0)$ **5 marks**

$$x + y = \frac{u}{\sqrt{2}} \qquad \textbf{... 1}$$

NEL($\vec{i}$): $\quad x - y = -e\left(\dfrac{u}{\sqrt{2}} - 0\right)$ **5 marks**

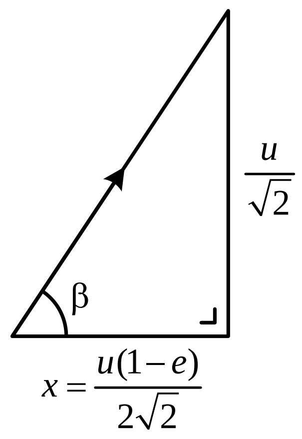
$$x - y = -\frac{eu}{\sqrt{2}} \qquad \textbf{... 2}$$

Adding **1** and **2**:

$$2x = \frac{u}{\sqrt{2}}(1-e)$$

$$x = \frac{u(1-e)}{2\sqrt{2}}$$ **5 marks**

Let β be the direction of A after impact.
Then



$$\tan\beta = \frac{\dfrac{u}{\sqrt{2}}}{\dfrac{u(1-e)}{2\sqrt{2}}}$$

$$= \frac{2}{1-e}$$ **5 marks**

If α is the angle through which A is deflected, then

$$\alpha = \beta - 45^\circ$$

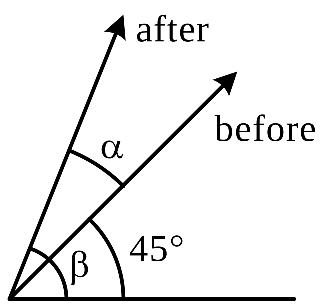
and $\quad \tan\alpha = \tan(\beta - 45^\circ)$

$$= \frac{\tan\beta - \tan 45^\circ}{1 + \tan\beta\tan 45^\circ}$$

$$= \frac{\dfrac{2}{1-e} - 1}{1 + \dfrac{2}{1-e}}$$

$$= \frac{2-(1-e)}{(1-e)+2}$$

$$= \frac{1+e}{3-e}.$$ **5 marks**



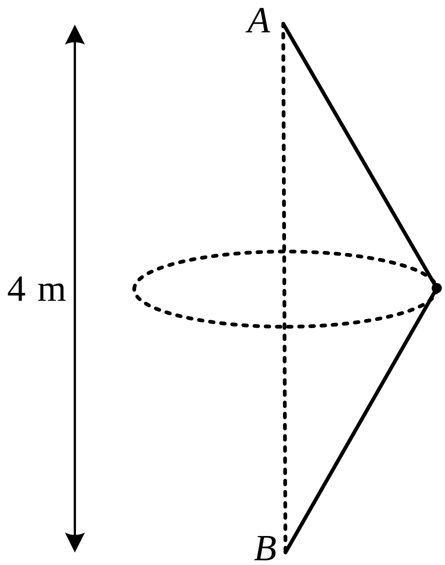
Question 6

(a) A particle of mass 5 kg is suspended from a fixed point by a light elastic string which hangs vertically. The elastic constant of the string is 500 N/m. The mass is pulled down a vertical distance of 20 cm from the equilibrium position and is then released from rest.

(i) Show that the particle moves with simple harmonic motion.

(ii) Find the speed and the acceleration of the mass $0 \cdot 1$ seconds after it is released from rest.

(b) A and B are two fixed pegs and A is 4 m vertically above B.
A mass *m* kg, connected to A and B by two light inextensible strings of equal length, is describing a horizontal circle with uniform angular velocity ω.
For what value of ω will the tension in the upper string be double the tension in the lower string?



Solution

(a) **(i)** *(First we must find the equilibrium position.)*

Let x be the extension when there is equilibrium. Then:

Newton's 2nd Law: $T = 5g$

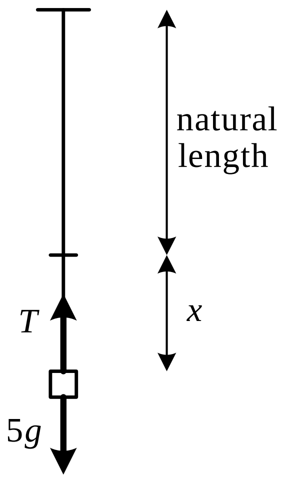
Hooke's Law: $T = k\,x$

$T = 500x$

Thus $500x = 5g$

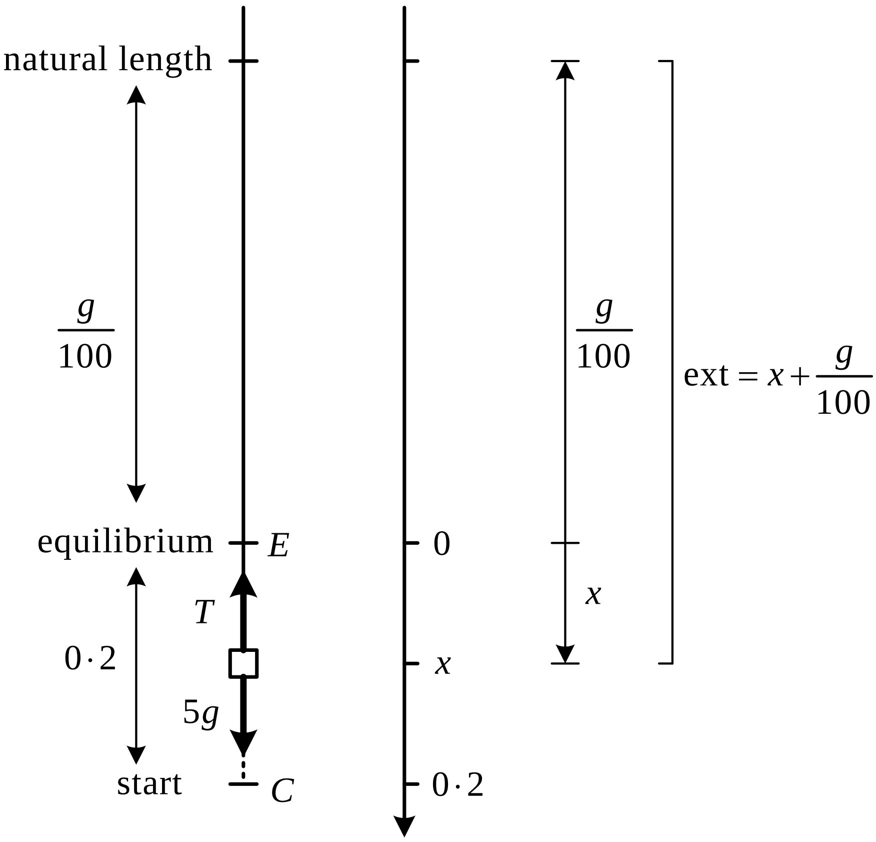
$$x = \frac{g}{100}$$

5 marks



(To establish SHM, we must consider the forces acting on the particle when it is in a general position. We consider the forces acting when the displacement is x below the equilibrium position.)

In general position x below the equilibrium position, the extension is $x+\dfrac{g}{100}$.



Hooke's Law:

$$T = 500\left(x+\frac{g}{100}\right) \qquad \textbf{... 1}$$

Newton's 2nd Law:

$$5\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 5g - T$$

$$5\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 5g - 500\left(x+\frac{g}{100}\right) \qquad \textbf{... by 1}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = g - 100x - g$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -100x$$

which is the equation of SHM about $x=0$ with

$\omega^2 = 100$

i.e. $\omega = 10.$ **<u>5 marks</u>**

(ii) *(As the particle starts from rest, the amplitude is the initial displacement from the equilibrium position.)*

$$a = 0 \cdot 2$$

(There are formulae for speed and acceleration in terms of x. So we should start by finding x when $t = 0 \cdot 1$. *As the particle starts from the positive edge, we use* $x = a\cos\omega t$*.)*

$t = 0 \cdot 1,\ x = ?:$

$$x = a\cos\omega t$$
$$x = 0 \cdot 2\cos(10)(0 \cdot 1)$$
$$x = 0 \cdot 2\cos 1$$
$$x = 0 \cdot 10806$$

<u>**5 marks**</u>

Then

$$v = \omega\sqrt{a^2 - x^2}$$
$$v = 10\sqrt{(0 \cdot 2)^2 - (0 \cdot 10806)^2}$$
$$v = 1 \cdot 68 \text{ m/s}$$

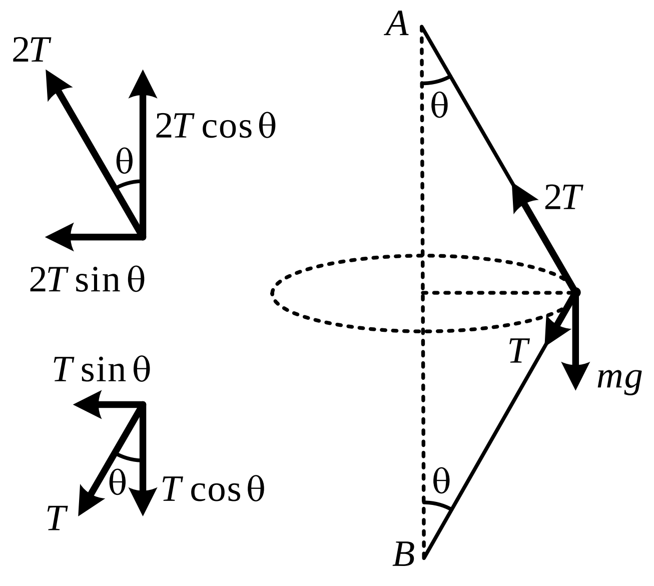
<u>**5 marks**</u>

and

$$\text{acc} = \omega^2 x$$
$$\text{acc} = (10)^2(0 \cdot 10806)$$
$$\text{acc} = 10 \cdot 8 \text{ m/s}^2.$$

<u>**5 marks**</u>

(b) *(We are told that the tension in the upper string is twice that in the lower string. If T is the tension in the lower string, then 2T is the tension in the upper string. Let* θ *be the angle between each string and the vertical.)*



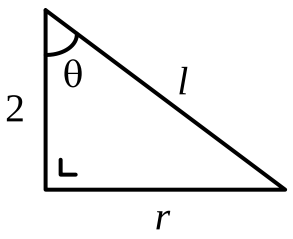
<u>**5 marks**</u>

(Let r be the length of the radius of the circle, and let l be the length of the string. We can draw a right-angled triangle showing these, and the third side which is of length 2.)

$\uparrow = \downarrow$:

$$2T\cos\theta = T\cos\theta + mg$$

<u>5 marks</u>



$$T\cos\theta = mg$$

$$T\left(\frac{2}{l}\right) = mg$$

$$T = \frac{mgl}{2}$$

<u>5 marks</u>

Circular motion:

$$2T\sin\theta + T\sin\theta = m\omega^2 r$$

<u>5 marks</u>

$$3\left(\frac{mgl}{2}\right)\sin\theta = m\omega^2 r$$

$$\frac{3gl}{2}\left(\frac{r}{l}\right) = \omega^2 r$$

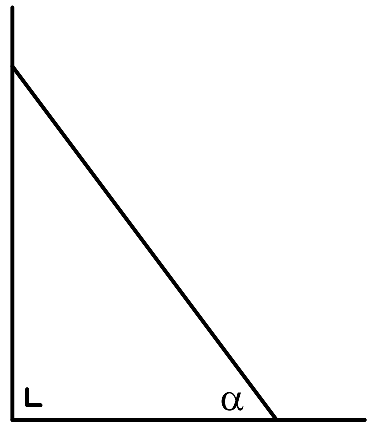
$$\omega^2 = \frac{3g}{2}$$

$$\omega = \sqrt{\frac{3g}{2}}\,.$$

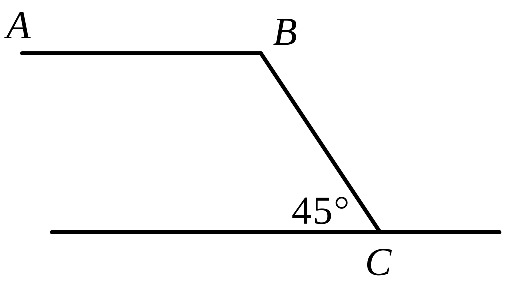
<u>5 marks</u>

Question 7

(a) One end of a uniform ladder, of weight *W*, rests against a rough vertical wall, and the other end rests on rough horizontal ground. The coefficient of friction at the ground is $\frac{1}{4}$ and at the wall is $\frac{1}{2}$.
The ladder makes an angle α with the horizontal and is in a vertical plane which is perpendicular to the wall.
The ladder is on the point of slipping. Find $\tan\alpha$.

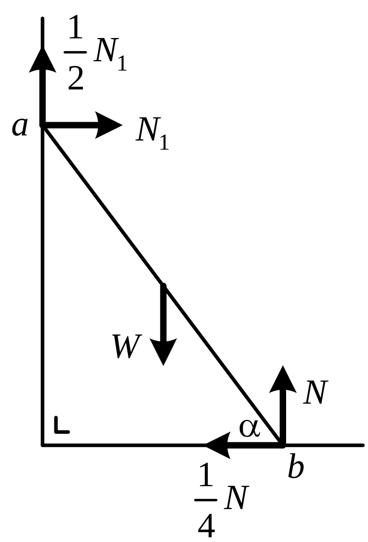


(b) Two equal uniform rods AB and BC smoothly jointed at B are in equilibrium with the end C resting on a rough horizontal surface. The end A is held above the surface.
The rod AB is horizontal and the rod BC is inclined at an angle of $45°$ to the horizontal.
If C is on the point of slipping find the coefficient of friction.



Solution

(a) *(We start by showing the forces acting on the ladder. Because the ladder is on the point of slipping, $F = \mu N$, both on the ground and at the wall.)*



$\uparrow = \downarrow: \quad N + \frac{1}{2}N_1 = W$ **... 1**

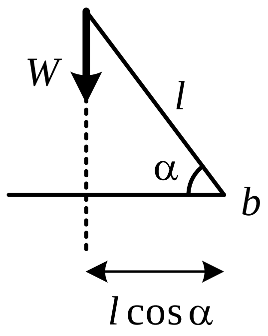
<u>5 marks</u>

$\leftarrow = \rightarrow: \quad \frac{1}{4}N = N_1$ **... 2**

<u>5 marks</u>

(We can take moments about any point we choose. Let 2l be the length of the ladder.)

Moments about b:



$$W.l\cos\alpha = N_1.2l\sin\alpha + \frac{1}{2}N_1.2l\cos\alpha$$

<u>5 marks</u>

$$W = 2N_1\tan\alpha + N_1$$
$$W = N_1(2\tan\alpha + 1)$$
$$N_1 = \frac{W}{2\tan\alpha + 1} \qquad \textbf{... 3}$$

Solving:

2, **3:** $N = \dfrac{4W}{2\tan\alpha + 1}$

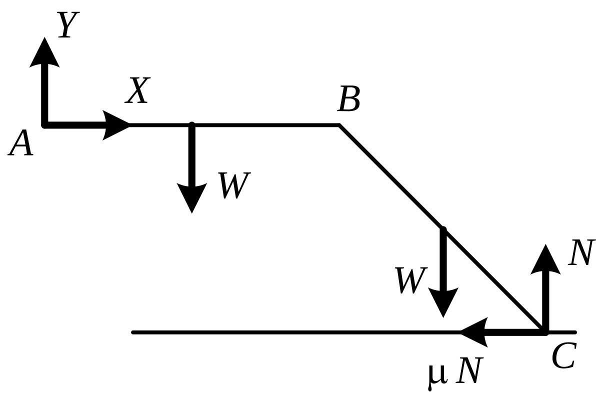
1: $\dfrac{4W}{2\tan\alpha+1} + \dfrac{1}{2}\left(\dfrac{W}{2\tan\alpha+1}\right) = W$ **<u>5 marks</u>**

$$4 + \frac{1}{2} = 2\tan\alpha + 1$$
$$\frac{7}{2} = 2\tan\alpha$$
$$\tan\alpha = \frac{7}{4}.$$

<u>5 marks</u>

(b) *(Let the length of each rod be 2l. Let the horizontal and vertical components of the supporting force at A be X and Y respectively.)*

Structure ABC:



$\uparrow = \downarrow: \quad Y + N = 2W$ **... 1**

<u>5 marks</u>

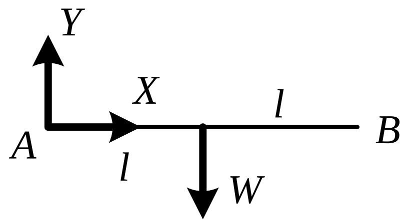
(We could put forces left equal to forces right, but we are going to try to avoid X. We could also take moments, but there are easier moments equations we can get.)

Rod AB:

Taking moments about B:

$W.l = Y.2l$ **<u>5 marks</u>**

$Y = \frac{1}{2}W$ **... 2**



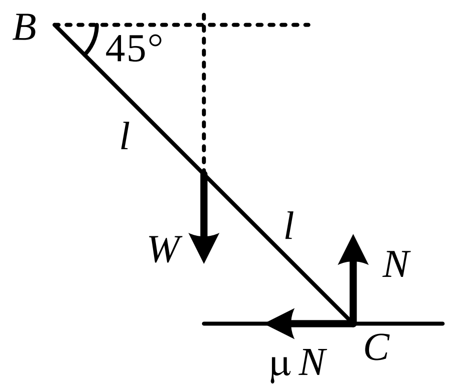
1, 2: $\frac{1}{2}W + N = 2W$

$N = \frac{3W}{2}$ **... 3**

<u>5 marks</u>

Rod BC:

Taking moments about B:



$W.l\cos 45° + \mu N.2l\sin 45° = N.2l\cos 45°$

<u>5 marks</u>

$W + 2\mu N = 2N$

3: $W + 2\mu\left(\frac{3W}{2}\right) = 2\left(\frac{3W}{2}\right)$

$1 + 3\mu = 3$

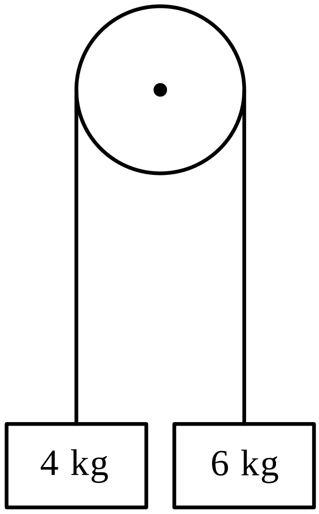
$3\mu = 2$

$\mu = \frac{2}{3}.$ **<u>5 marks</u>**

Question 8

(a) Prove that the moment of inertia of a uniform circular disc, of mass *m*, and radius *r*, about an axis through its centre perpendicular to its plane is $\frac{1}{2}mr^2$.

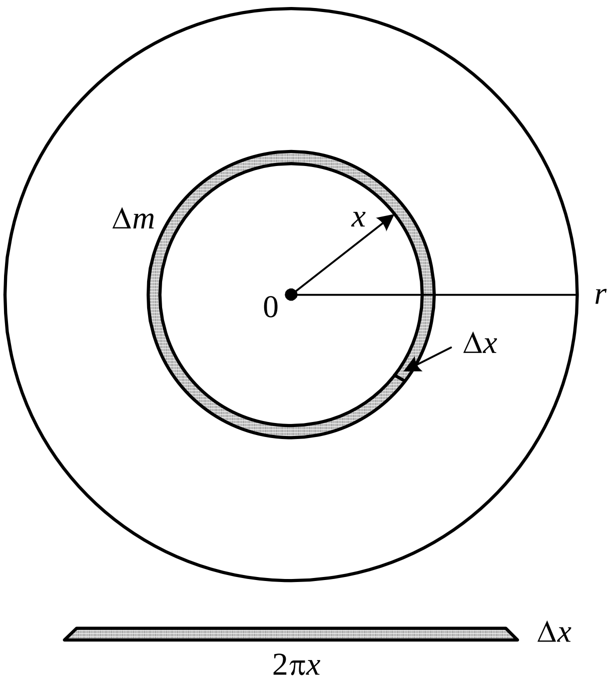
(b) Masses of 4 kg and 6 kg are suspended from the ends of a light inextensible string which passes over a pulley. The axis of rotation of the pulley is horizontal, perpendicular to the pulley, and passes through the centre of the pulley. The moment of inertia of the pulley is $0 \cdot 08 \text{ kg m}^2$ and its radius is 20 cm. The particles are released from rest and move vertically. When each mass has acquired a speed of 1 m/s, find



(i) the common acceleration of the masses,
(ii) the tensions in the vertical portions of the string.

Solution

(a) *(The diagram below shows what is going on.)*



Let ρ be the density (mass per unit area) of the disc. Then

$$\rho = \frac{m}{\pi r^2}$$

(When the narrow strip is opened out, it is approximately a rectangle with width $2\pi x$ and height Δx.)

$$\rho = \frac{\Delta m}{2\pi x \Delta x}$$

Thus $\Delta m = 2\pi\rho x \Delta x$

Then $I = \sum_{\Delta m} x^2 \, \Delta m$ **5 marks**

$= \sum_{\Delta x} x^2 \, 2\pi\rho x \, \Delta x$

$= \int_0^r 2\pi\rho x^3 \, \mathrm{d}x$ **5 marks**

$= \left[\frac{2\pi\rho}{4} x^4 \right]_0^r$ **5 marks**

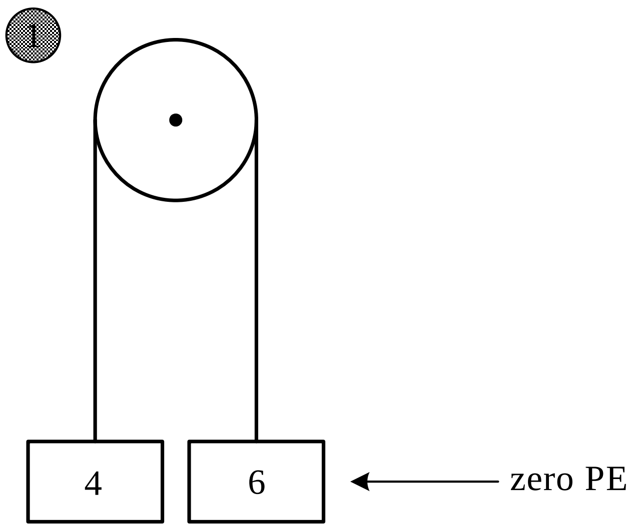
$= \frac{\pi\rho r^4}{2} = \frac{\pi}{2}\left(\frac{m}{\pi r^2}\right) r^4 = \frac{1}{2} m r^2.$ **5 marks**

(b) (i) *(Let Position 1 be the starting position where all objects are at rest. Let the zero PE levels for the two particles be this position.)*

$KE_1 = 0$

and

$PE_1 = 0$



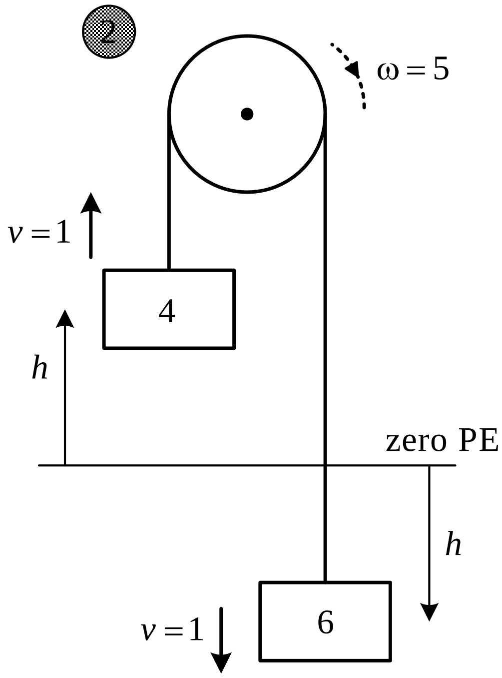
(Let Position 2 be where each particle has a speed of 1 m/s. To find the KE here, we will need the KE of the pulley. For this, we will need ω, its angular velocity. The radius of the pulley is $0 \cdot 2$ m.)

$v = r\,\omega$

$1 = (0 \cdot 2)\omega$

$\omega = 5$

(The KE in Position 2 is the sum of the KEs of the pulley and the two particles. We use $\frac{1}{2}I\omega^2$ for the pulley and $\frac{1}{2}mv^2$ for each particle.)



$$
\begin{aligned}
KE_2 &= \frac{1}{2}(0\cdot 08)(5)^2 \\
&\quad + \frac{1}{2}(4)(1)^2 \\
&\quad + \frac{1}{2}(6)(1)^2 \\
&= 1+2+3 \\
&= 6
\end{aligned}
$$

5 marks

(The PE of the 4 kg particle is positive while that of the 6 kg particle is negative. We use $\pm mgh$ for each.)

$$
\begin{aligned}
PE_2 &= (4)g(h)-(6)g(h) \\
&= -2gh
\end{aligned}
$$

5 marks

(Now we apply the Principle of Conservation of Energy, PCE, i.e. the total mechanical energy is constant.)

PCE:
$$
\begin{aligned}
KE_2 + PE_2 &= KE_1 + PE_1 \\
6-2gh &= 0+0 \\
6 &= 2gh \\
h &= \frac{3}{g}
\end{aligned}
$$

5 marks

(Because all the forces are constant, we can use the linear motion equations to find the acceleration.)

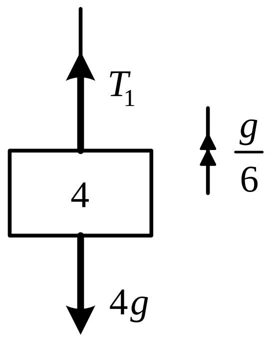
$u = 0$
$v = 1$
$a = ?$
$s = \frac{3}{g}$
$t = -$

Then

$$v^2 = u^2 + 2as$$
$$(1)^2 = (0)^2 + 2a\left(\frac{3}{g}\right)$$
$$1 = \frac{6a}{g}$$
$$a = \frac{g}{6} \text{ m/s}^2.$$
5 marks

(ii) *(Due to the inertia of the pulley, the tension forces in the string on each side of the pulley are not the same. Let T_1 be the tension on the left and T_2 be the tension on the right.)*

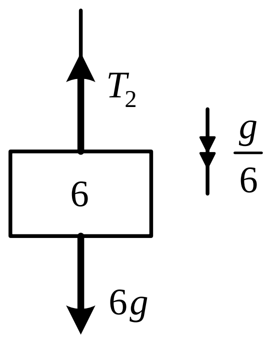
4 kg mass:



Newton's 2nd Law:

$$4\left(\frac{g}{6}\right) = T_1 - 4g$$
$$\frac{2g}{3} = T_1 - 4g$$
$$T_1 = \frac{14g}{3}$$
$$= 45 \cdot 7 \text{ N}$$
5 marks

6 kg mass:

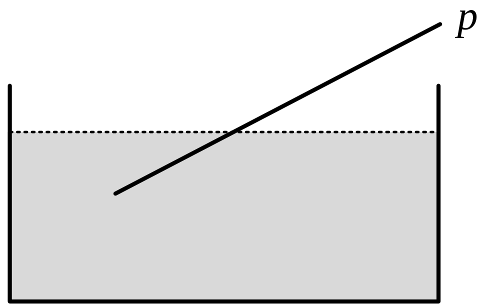


Newton's 2nd Law:

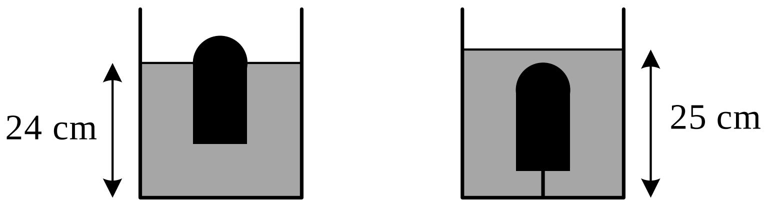
$$6\left(\frac{g}{6}\right) = 6g - T_2$$
$$T_2 = 6g - g$$
$$T_2 = 5g$$
$$= 49 \text{ N.}$$
5 marks

Question 9

(a) A uniform rod, of length 2 m and relative density $\frac{7}{9}$, is pivoted at one end p and is free to move about a horizontal axis through p. The other end of the rod is immersed in water. The rod is in equilibrium and is inclined to the vertical as shown in the diagram. Find the length of the immersed part of the rod.



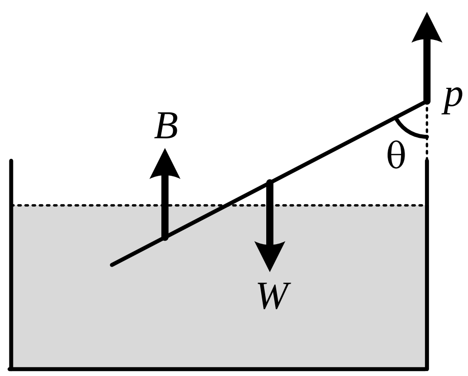
(b) A cylinder contains water to a height of 20 cm. A solid body of mass $0 \cdot 06$ kg is placed in the cylinder. It floats and the water level rises to 24 cm.



The body is then completely submerged and tied by a string to the bottom of the cylinder. The water level rises to 25 cm.
Find
(i) the relative density of the body
(ii) the tension in the string
(iii) the radius of the cylinder.

Solution

(a) *(Let the length of the submerged part of the rod be l, and let W be the weight of the rod. Let the rod make an angle of θ with the vertical. Let B be the buoyancy.)*



Fraction of rod which is submerged
$$= \frac{l}{2}.$$

Weight of submerged part of rod
$$= \frac{l}{2}W = \frac{Wl}{2}.$$

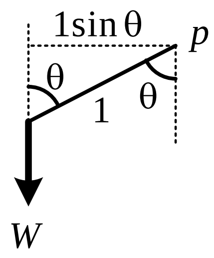
Then $$B = \frac{\text{weight of submerged part}}{s}$$

<u>5 marks</u> (for diagram)

$$B = \frac{\frac{Wl}{2}}{\frac{7}{9}}$$

$$B = \frac{9lW}{14} \qquad \textbf{... 1}$$

<u>5 marks</u>



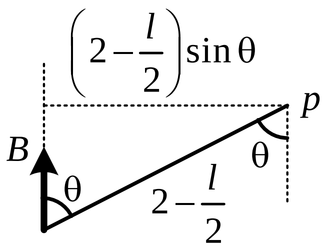
Taking moments about p:

$$W.(1\sin\theta) = B.\left(\left(2-\frac{l}{2}\right)\sin\theta\right)$$

<u>5 marks</u>

$$W = B\left(2-\frac{l}{2}\right) \qquad \textbf{... 2}$$

Solving **1**, **2:**



$$W = \frac{9lW}{14}\left(2-\frac{l}{2}\right)$$

$$14 = 9l\left(2-\frac{l}{2}\right)$$

$$28 = 9l(4-l)$$

$$28 = 36l - 9l^2$$

$$9l^2 - 36l + 28 = 0$$

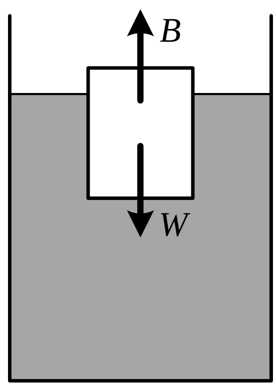
$$l = \frac{36 \pm \sqrt{36^2 - 4(9)(28)}}{18}$$

$$l = \frac{36 \pm 12\sqrt{2}}{18}$$

$$l = 2\cdot 94 \quad \text{or} \quad l = 1\cdot 06$$

As $l < 2$, $l = 1\cdot 06$ m. **<u>5 marks</u>**

(b) **(i)** *(When floating, the water rises 4 cm. When fully submerged, the water level rises 5 cm. Thus, when floating, $\frac{4}{5}$ of the body is submerged.)*



Let V be the volume of the object.

When floating,

$$B = \left(\frac{4}{5}V\right)(1000)g$$

and

$$W = V(1000)sg$$

In equilibrium:

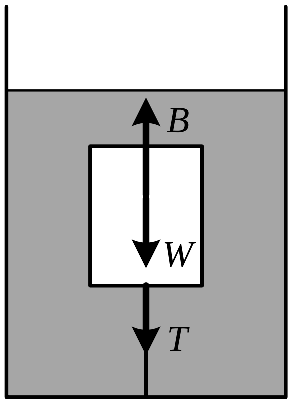
$$B = W$$

$$\left(\frac{4}{5}V\right)(1000)g = V(1000)sg$$

$$\frac{4}{5} = s$$

$$s = 0 \cdot 8$$ **<u>5 marks</u>**

(ii) When tied to the bottom of the cylinder:



$$B = \frac{W}{s}$$

$$B = \frac{mg}{s}$$

$$B = \frac{0 \cdot 06g}{0 \cdot 8}$$

$$B = 0 \cdot 075g$$ **<u>5 marks</u>**

In equilibrium:

$$B = T + W$$ **<u>5 marks</u>**

$$0 \cdot 075g = T + 0 \cdot 06g$$

$$T = 0 \cdot 015g$$

$$T = 0 \cdot 147 \text{ N}$$ **<u>5 marks</u>**

(iii) *(To find the radius, we need to find the cross-sectional area, A, of the cylinder. The volume of the object is the same as the volume of the displaced liquid, when fully submerged.)*

Volume of displaced liquid $= 0 \cdot 05A$
Volume of object $= 0 \cdot 05A$
Weight of object $= 1000sVg$
$= 1000(0 \cdot 8)(0 \cdot 05A)g$
$= 40Ag$

(But we know that the weight of the object is $0 \cdot 06g$*.)*

Thus $40Ag = 0 \cdot 06g$

$$A = \frac{0 \cdot 06}{40}$$

$$A = 0 \cdot 0015$$ **<u>5 marks</u>**

(If r is the radius of the cylinder, then $A = \pi r^2$*.)*

$\pi r^2 = 0 \cdot 0015$

$r^2 = 0 \cdot 00047746$

$r = 0 \cdot 0218$ m

$r = 2 \cdot 18$ cm. **5 marks**

Question 10

(a) If

$$x^2 y\frac{dy}{dx} + y\frac{dy}{dx} = 1$$

and $y = 0$ when $x = 0$, find the value of x when $y = \sqrt{\frac{\pi}{2}}$.

(b) A train of mass 200 tonnes moves along a straight level track against a resistance of $400v^2$, where v m/s is the speed of the train.
The engine exerts a constant power of P kW.
The acceleration of the train is $\dfrac{8000 - v^3}{500v}$.

(i) Find the value of P.

(ii) The train travels a distance of $69 \cdot 07$ m while its speed increases from 10 m/s to v_1 m/s. Find the value of v_1.

Solution

(a)

$$x^2 y\frac{dy}{dx} + y\frac{dy}{dx} = 1$$

$$y\frac{dy}{dx}\left(x^2+1\right) = 1$$

$$y\frac{dy}{dx} = \frac{1}{x^2+1}$$

$$y\,dy = \frac{1}{x^2+1}dx$$

$$\int y\,dy = \int \frac{dx}{x^2+1}$$ **<u>5 marks</u>**

$$\frac{1}{2}y^2 = \tan^{-1} x + c$$ **<u>5 marks</u>**

(Now we use the initial conditions to find c.)

$x = 0,\ y = 0:$ $\quad 0 = 0 + c$

$c = 0$ **<u>5 marks</u>**

(Now we can get the unique solution.)

$$\frac{1}{2}y^2 = \tan^{-1} x$$

$y = \sqrt{\frac{\pi}{2}}:\quad \frac{1}{2}\left(\frac{\pi}{2}\right) = \tan^{-1} x$

$$\frac{\pi}{4} = \tan^{-1} x$$

$$\tan\frac{\pi}{4} = x$$

$$x = 1.$$ **<u>5 marks</u>**

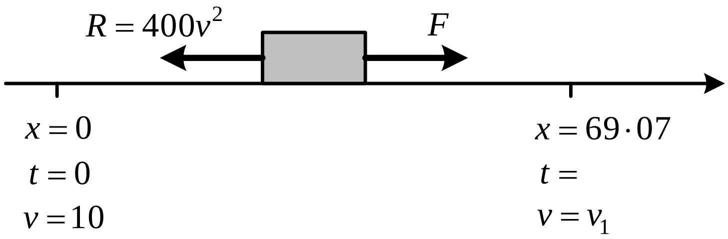
(b) (i) *(It will be necessary to put in some effort to obtain the equation of motion. The power of the train engine is P kW, i.e.* $1000P$ *W. As*

Power = Force × Velocity,

we can obtain an expression for the force, F, in terms of the velocity, v. As v is variable, the force will be variable. Also, the mass of the train is 200000 kg.*)*

Let F be the force of the engine. Then

$$\text{Power} = \text{Force} \times \text{Velocity}$$
$$1000P = F \times v$$
$$F = \frac{1000P}{v}$$



(Now we can construct the equation of motion, through Newton's 2nd Law.)

Newton's 2nd Law:

$$200000\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = F - 400v^2$$

$$200000\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{1000P}{v} - 400v^2$$

$$1000\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{5P}{v} - 2v^3$$

$$1000\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{5P - 2v^3}{v}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{5P - 2v^3}{1000v}$$

(But we are told that the acceleration is $\frac{8000 - v^3}{500v}$*.)*

Thus $\dfrac{5P-2v^3}{1000v}=\dfrac{8000-v^3}{500v}$

$5P-2v^3=2(8000-v^3)$

$5P-2v^3=16000-2v^3$

$P=3200.$ **5 marks**

(ii) *(We are asked to find the velocity at a given point, i.e. at a value of x. Thus we only need one integration. Because the acceleration is in terms of v only, and we have all the necessary initial conditions, we will use* $v\dfrac{dv}{dx}$ *for the acceleration.*

Because we only want a value, i.e. the value of v_1*, we will use definite integrals. This time we won't change the limits.)*

$v\dfrac{dv}{dx}=\dfrac{8000-v^3}{500v}$ **5 marks**

$\dfrac{v^2\,dv}{8000-v^3}=\dfrac{1}{500}dx$

$\displaystyle\int_{10}^{v_1}\frac{v^2\,dv}{8000-v^3}=\frac{1}{500}\int_0^{69\cdot07}dx$ **... 1** **5 marks**

Let $u=8000-v^3$

$du=-3v^2\,dv$

$-\dfrac{1}{3}du=v^2\,dv$

Then

$\displaystyle\int\frac{v^2\,dv}{8000-v^3}=-\frac{1}{3}\int\frac{du}{u}$

$=-\dfrac{1}{3}\ln|u|$

$=-\dfrac{1}{3}\ln\left|8000-v^3\right|$

1: $-\dfrac{1}{3}\Big[\ln\left|8000-v^3\right|\Big]_{10}^{v_1}=\dfrac{1}{500}\Big[x\Big]_0^{69\cdot07}$ **5 marks**

$\ln\left|8000-v_1^3\right|-\ln 7000=\dfrac{-3}{500}(69\cdot07)$ **5 marks**

$\log_e\dfrac{\left|8000-v_1^3\right|}{7000}=-0\cdot41442$

$\dfrac{8000-v_1^3}{7000}=e^{-0\cdot41442}$

$8000-v_1^3=4625\cdot0637$

$v_1^3=3374\cdot9363$

$v_1=15.$ **5 marks**