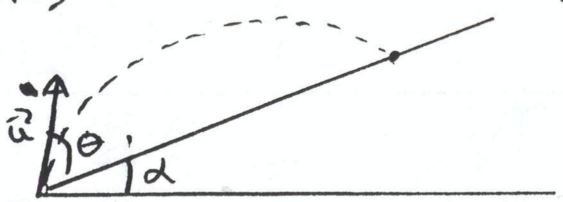
PROJECTILES.

(7) (H 1978)



$\vec{g}$: $-g\cos\alpha\,\vec{j}$, $-g\sin\alpha\,\vec{i}$.

$$\vec{u} = u\cos\theta\,\vec{i} + u\sin\theta\,\vec{j}.$$
$$\vec{g} = -g\sin\alpha\,\vec{i} - g\cos\alpha\,\vec{j}.$$

At any time t:

$$\vec{r}(t) = \left(u\cos\theta t - \tfrac{g}{2}\sin\alpha t^2\right)\vec{i} + \left(u\sin\theta t - \tfrac{g}{2}\cos\alpha t^2\right)\vec{j}$$

$$\vec{v}(t) = (u\cos\theta - g\sin\alpha t)\vec{i} + (u\sin\theta - g\cos\alpha t)\vec{j}.$$

Time taken to reach max height:

$\vec{j}$ cpt of $\vec{v} = 0 \Rightarrow u\sin\theta - g\cos\alpha t = 0$

$$\Rightarrow t = \frac{u\sin\theta}{g\cos\alpha}. \quad (*)$$

Time taken in flight:

$\vec{j}$ cpt of $\vec{r} = 0$:

$$u\sin\theta t - \tfrac{g}{2}\cos\alpha t^2 = 0.$$

$$\therefore t = 0 \text{ OR } t = \frac{2u\sin\theta}{g\cos\alpha}. \quad (\Delta)$$

$*$ and $(\Delta) \Rightarrow$

$\therefore$ Time in flight $= 2$(time to reach max height)

When $t = \dfrac{u\sin\theta}{g\cos\alpha}$ (particle at max height)

$\vec{i}$ cpt of $\vec{r}$ is.

$$u\cos\theta\left(\frac{u\sin\theta}{g\cos\alpha}\right) - \frac{g}{2}\sin\alpha\left(\frac{u\sin\theta}{g\cos\alpha}\right)^2.$$

But told that this must be same as $\frac{2}{3}$(Range)

$$= \frac{2}{3}\left(u\cos\theta\left[\frac{2u\sin\theta}{g\cos\alpha}\right] - \frac{g}{2}\sin\alpha\left[\frac{2u\sin\theta}{g\cos\alpha}\right]^2\right)$$

$$\therefore u\cos\theta\left(\frac{u\sin\theta}{g\cos\alpha}\right) - \frac{g}{2}\sin\alpha\left(\frac{u\sin\theta}{g\cos\alpha}\right)^2 = \frac{2}{3}\left(u\cos\theta\left[\frac{2u\sin\theta}{g\cos\alpha}\right] - \frac{g}{2}\sin\alpha\left[\frac{2u\sin\theta}{g\cos\alpha}\right]^2\right)$$

$$\therefore \frac{u^2\cos\theta\sin\theta}{g\cos\alpha} - \frac{u^2\sin^2\theta\sin\alpha}{2g\cos^2\alpha} = \frac{4}{3}\frac{u^2\cos\theta\sin\theta}{g\cos\alpha} - \frac{g}{3}\frac{\sin\alpha\, 4u^2\sin^2\theta}{g^2\cos^2\alpha}$$

$$\therefore -\frac{1}{3}\frac{u^2\cos\theta\sin\theta}{g\cos\alpha} = \frac{u^2\sin^2\theta\sin\alpha}{2g\cos^2\alpha} - \frac{4u^2\sin^2\theta\sin\alpha}{3g\cos^2\alpha}.$$

$$\therefore -\frac{1}{3}\cos\theta\sin\theta = \frac{\sin^2\theta\sin\alpha}{2\cos\alpha} - \frac{4}{3}\frac{\sin^2\theta\sin\alpha}{\cos\alpha}.$$

$$\therefore -\frac{1}{3}\cos\theta\sin\theta = \frac{\sin^2\theta\tan\alpha}{2} - \frac{4}{3}\sin^2\theta\tan\alpha$$

$$\therefore -\frac{1}{3}\cos\theta\sin\theta = \frac{3\sin^2\theta\tan\alpha - 8\sin^2\theta\tan\alpha}{6}$$

$$\therefore -\frac{1}{3}\cos\theta\sin\theta = -\frac{5}{6}\sin^2\theta\tan\alpha.$$

$$\therefore \frac{2}{5}\cos\theta = \sin\theta\tan\alpha.$$

$$\therefore \frac{2}{5} = \tan\theta\tan\alpha \qquad \text{q.e.d.}$$