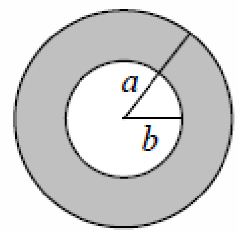
2010 : Moments of Inertia Question

8. **(a)** Prove that the moment of inertia of a uniform circular disc, of mass m and radius r, about an axis through its centre perpendicular to its plane is $\frac{1}{2}mr^2$.

(b) An annulus is created when a central hole of radius b is removed from a uniform circular disc of radius a.

The mass of the annulus (shaded area) is M.



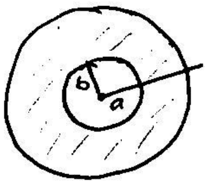
(i) Show that the moment of inertia of the annulus about an axis through its centre and perpendicular to its plane is $\frac{M\left(a^2+b^2\right)}{2}$.

(ii) The annulus rolls, from rest, down an incline of 30°. Find its angular velocity, in terms of g, a and b, when it has rolled a distance $\frac{a}{2}$.

<u>2010</u>

(a) PROOF

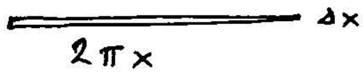
(b)(i) VERY SIMILAR TO THE DISK PROOF:



$$\rho = \frac{\text{MASS}}{\text{AREA}}$$

$$\rho = \frac{M}{\pi a^2 - \pi b^2} \Rightarrow \rho = \frac{M}{\pi(a^2 - b^2)}$$

DIVIDE INTO STRIPS OF WIDTH Δx, EACH A DISTANCE x FROM THE CENTRE.



so

AREA $= 2\pi x(\Delta x)$

$$\rho = \frac{\text{MASS}}{\text{AREA}} \Rightarrow \rho = \frac{\Delta m}{2\pi x \Delta x} \Rightarrow 2\pi\rho x \Delta x = \Delta m$$

$$I = \sum \Delta m . r^2$$

$$I = \sum 2\pi\rho x \Delta x . x^2$$

$$I = \int_b^a 2\pi\rho x^3 . dx$$

5

$$I = \left[\frac{2\pi\rho x^4}{4}\right]_b^a$$

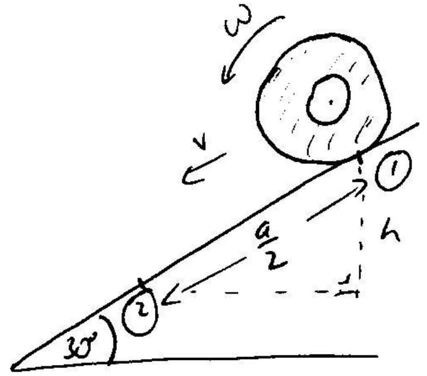
$$I = \frac{\pi\rho a^4 - \pi\rho b^4}{2}$$

$$I = \pi\rho\left[\frac{a^4 - b^4}{2}\right]$$

$$I = \pi \left[\frac{m}{\pi(a^2-b^2)}\right] \cdot \left[\frac{a^4-b^4}{2}\right] \quad (5)$$

$$I = \frac{m}{\cancel{a^2-b^2}} \cdot \frac{(a^2+b^2)\cancel{(a^2-b^2)}}{2} \Rightarrow I = \frac{m(a^2+b^2)}{2} \quad (5)$$

(ii)



Energy at (1): P.E. + K.E.$_{\text{linear}}$ + K.E.$_{\text{angular}}$

$$mg\bar{h} + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \quad (5)$$

$$mg\left(\frac{a}{2}\sin 30^\circ\right) + \frac{1}{2}(m)(0)^2 + \frac{1}{2}I(0)^2$$

$$\boxed{\frac{mga}{4}} \quad (1)$$

Energy at (2): P.E. + K.E.$_{\text{lin.}}$ + K.E.$_{\text{ang.}}$

$$mg\bar{h} + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$mg(0) + \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{m(a^2+b^2)}{2}\right)\omega^2$$

$* \; v = r\omega$

$v = a\omega$

$$0 + \frac{1}{2}m(a\omega)^2 + \frac{m(a^2+b^2)}{4}\omega^2$$

$$\boxed{\frac{ma^2\omega^2}{2} + \frac{m(a^2+b^2)\omega^2}{4}} \quad 2$$

$$1 = 2$$

$$\frac{mga}{4} = \frac{ma^2\omega^2}{2} + \frac{m(a^2+b^2)\omega^2}{4} \quad 5$$

$(\times 4)$

$$ga = 2a^2\omega^2 + (a^2+b^2)\omega^2$$

$$ga = 2a^2\omega^2 + a^2\omega^2 + b^2\omega^2$$

$$ga = 3a^2\omega^2 + b^2\omega^2$$

$$ga = (3a^2+b^2)\omega^2$$

$$\frac{ga}{3a^2+b^2} = \omega^2$$

$$\sqrt{\frac{ga}{3a^2+b^2}} = \omega \quad 5$$